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LETTER TO THE EDITOR

Garden of Eden states in a traffic model revisited

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Abstract. A modified mean-field theory is framed to adjoin two successful mean-field theories: cluster approximation and paradisical mean-field theory. The results do not give an improvement in describing the fundamental diagram. We point out that the success of paradisical mean-field theory is accidental and cannot be improved systematically as the cluster approximation can.

1. Introduction

The cellular automaton description of traffic flow has attracted much attention recently [1]. Instead of differential equations, the underlying dynamics is governed by a few update rules [2]. They are suitable for large-scale computer calculations and have been used for real-time simulations of urban traffic in various cities [3]. Numerical works in various applications have been reported [4]. In contrast, we know very little about the analytical properties. The difficulty of analytical descriptions is partly owing to the lack of a Hamiltonian; thus standard methods in statistical mechanics are not applicable. Complementary to numerical works, analytical descriptions can provide better insights into the models and help greatly to reduce the need for computer resources.

Various analytical approaches have been developed to describe the fundamental diagram, i.e. the relation between flow and density in the stationary states. The mean-field theory based on a microscopic description is the simplest one. As correlations between cells are completely neglected, the flow is underestimated considerably. More recently, different methods of improvement have been proposed. Especially, both the cluster approximation and the paradisical mean-field theory are able to obtain the exact results in the limiting case of $v_{\max} = 1$.

In the cluster approximation, the short-range correlations between the cells are systematically taken into account [5, 6]. A cluster of n neighbouring cells is treated exactly in the n-cluster approximation. The cluster is then coupled to the rest of the system self-consistently by overlapping $(n - 1)$ cells with another cluster. In the limiting case of $n = 1$, this approach reduces to the simple mean-field theory. The exact results of $v_{\max} = 1$ can be reproduced in the two-cluster approximation.

The paradisical mean-field theory is a simple mean-field theory applied in a reduced configuration space. In the parallel update, not all states of the configuration space can be reached by the dynamics. These dynamically forbidden states are called Garden of Eden (GoE) states or *paradisical states* [7], which are eliminated deliberately from the paradisical mean-field theory. The exact result of $v_{\max} = 1$ can be readily reproduced.

However, neither the two-cluster approximation nor paradisical mean-field theory is able to obtain exact results in the case of $v_{\max} = 2$, though considerable improvement over the simple mean-field results has been achieved in both approaches. Inclusion of the short-range correlations and elimination of the dynamically forbidden states are both important. However, these two issues have been considered separately in previous works. In this Letter, we combine these two approaches in the hope of obtaining a better description.

2. Traffic model

In the basic model [2], a single-lane highway is divided into L cells. Each cell can be either empty or occupied by a car with an integer speed $v \in \{0, 1, \ldots, v_{\max}\}$, where $v_{\max}$ is the speed limit. With periodic boundary condition, the number of cars is conserved. At each time step, the configuration of N cars is updated by the following four rules, which are applied in parallel to all cars. The first rule (R1) is the acceleration. If the speed of a car is lower than $v_{\max}$, the speed is advanced by one. The second rule (R2) is the slowing down due to other cars. If a car has d empty cells in front of it and a speed larger than d, the speed is reduced to d. The third rule (R3) is the randomization, which introduces a noise to simulate the stochastic driving behaviour. The speed of a moving car ($v \geqslant 1$) is decreased by one with a braking probability p. In the fourth rule (R4), the position of a car is shifted by its speed v. Iterations over these simple rules already give realistic results. The model contains three parameters: the maximum speed $v_{\max}$, the braking probability p and the average density $\rho = N/L$.

3. Modified mean-field theory

In this section, we present a self-consistent approach to combine the n-cluster approximation with the paradisical mean-field theory. The GoE states in the two-cluster approximation are eliminated and a new normalization constant is introduced. We briefly review the results for $v_{\max} = 2$ in the two-cluster approximation. In order to simplify the description, a slightly different update ordering (R2–R3–R4–R1) is suggested, i.e. one looks at the system after the acceleration rule. Then there are no cars with $v = 0$ and effectively the number of variables is reduced. In the two-cluster approximation, the nearest-neighbour correlations are considered. There are nine variables in describing the configurations of two nearest-neighbouring cells: P_{ij} with $i, j \in \{x, 1, 2\}$, where x denotes an empty cell and the numbers correspond to the speeds of the cars. We note that the benefit of ordering (R2–R3–R4–R1) is in dealing with a smaller number of variables, while the GoE states can be easily analysed in the ordering (R1–R2–R3–R4). To distinguish between these two orderings in the following text, we use $\langle\langle\cdots\rangle\rangle$ to denote the configurations in the ordering (R2–R3–R4–R1) and $\langle\cdots\rangle$ in the ordering (R1–R2–R3–R4). It is interesting to note that the parallel update implies that two of the nine variables vanish, i.e. $P_{12} = P_{22} = 0$. The configurations $\langle\langle 12\rangle\rangle$ correspond to $\langle 01\rangle$ and $\langle 02\rangle$. Similarly, the configurations $\langle\langle 22\rangle\rangle$ correspond to $\langle 11\rangle$, $\langle 12\rangle$, $\langle 21\rangle$ and $\langle 22\rangle$. These are GoE states, since the moving car must leave an empty cell behind. As the nearest-neighbouring cells are treated exactly in the two-cluster approximation, these nearest-neighbouring GoE states are excluded automatically. The normalization of probabilities gives

$$P_{xx} + P_{x1} + P_{x2} + P_{1x} + P_{11} + P_{2x} + P_{21} = 1. \tag{1}$$

The conservation of the number of cars gives

$$P_{x1} + P_{x2} + P_{1x} + 2P_{11} + P_{2x} + 2P_{21} = 2\rho. \tag{2}$$

The equilibrium probabilities are determined by the dynamics of the update rules. The equations for the equilibrium probabilities can be obtained by the combination of conditional probabilities. For example, the equation for P_{1x} reads

$$P_{1x} = pP_{1x} + P_{2x}\left[\frac{pP_{x1} + pP_{x2}}{P_{xx} + P_{x1} + P_{x2}}\right] + P_{11}\left[\frac{(1-p)P_{1x}}{P_{1x} + P_{11}}\right] + P_{21}\left[\frac{(1-p)P_{1x}}{P_{1x} + P_{11}}\right]. \tag{3}$$

The first term in the right-hand side describes the probability for the configuration $\langle\langle 1x\rangle\rangle$ to remain unchanged in the next time step. The other three terms correspond to the probabilities for the configurations $\langle\langle 2x\rangle\rangle$, $\langle\langle 11\rangle\rangle$ and $\langle\langle 21\rangle\rangle$, respectively, to change into $\langle\langle 1x\rangle\rangle$ in the next time step. As prescribed in the update rules, the configuration $\langle\langle 1x\rangle\rangle$ can only arise from these four configurations. Similarly, there are six more equations for the equilibrium values of the other six variables: P_{xx}, P_{x1}, P_{x2}, P_{11}, P_{2x} and P_{21}, which are not shown. These equations are not linearly independent of each other. However, together with equations (1) and (2), they provide a unique solution for P_{ij}. The flow can be written as

$$J = (1-p)P_{1x} + P_{2x}\left[\frac{(2-p)P_{xx} + (1-p)P_{x1} + (1-p)P_{x2}}{P_{xx} + P_{x1} + P_{x2}}\right]. \tag{4}$$

In the two-cluster approximation, the GoE states beyond the nearest-neighbouring cells are still included. For example, configurations $\langle 1x2\rangle$ and $\langle 2x2\rangle$ are also GoE states, which become parts of the configurations $\langle\langle 2x2\rangle\rangle$. These states are assigned with a probability

$$P_{2x}\left[\frac{P_{x2}}{P_{xx} + P_{x1} + P_{x2}}\right] \tag{5}$$

and appear in the second term of equation (3). However, we cannot just eliminate this term as done in the paradisical mean-field theory, for the states $\langle\langle 2x2\rangle\rangle$ also include the states $\langle 1x1\rangle$ and $\langle 2x1\rangle$, which are not GoE states. Thus the states $\langle\langle 2x2\rangle\rangle$ can only be partly eliminated. Only those configurations where the leading car has a speed of 2 before applying the acceleration rule are the GoE states. Thus we have to distinguish between $v = 1$ and $v = 2$, which are mixed up after applying the acceleration rule. We introduce a new parameter $\mathcal{W}$ to describe the weighting of GoE states in the configuration $\langle\langle 2x2\rangle\rangle$. Then the equation for P_{1x} should be replaced by

$$P_{1x} = \mathcal{N}\left\{pP_{1x} + P_{2x}\left[\frac{pP_{x1} + pP_{x2}(1-\mathcal{W})}{P_{xx} + P_{x1} + P_{x2}}\right] + P_{11}\left[\frac{(1-p)P_{1x}}{P_{1x} + P_{11}}\right] + P_{21}\left[\frac{(1-p)P_{1x}}{P_{1x} + P_{11}}\right]\right\}. \tag{6}$$

As the configuration space is reduced, a new normalization $\mathcal{N}$ is necessary. The weighting $\mathcal{W}$ can be explicitly expressed as

$$\mathcal{W} = \mathcal{N}\left[\frac{P_{2x}}{P_{2x} + P_{21}}\right]\left[\frac{P_{xx}}{P_{xx} + P_{x1} + P_{x2}}\right](1-p) \tag{7}$$

where the case of $v = 2$ before applying the acceleration rule can be related to a configuration $\langle\langle 2xx\rangle\rangle$. The normalization constant $\mathcal{N}$ is inserted for self-consistency. The same factor can also be used to eliminate the GoE states in the configurations $\langle\langle 1x2\rangle\rangle$ and $\langle\langle 1xx2\rangle\rangle$, in which $\langle 0x2\rangle$ and $\langle 0xx2\rangle$ are GoE states, while $\langle 0x1\rangle$ and $\langle 0xx1\rangle$ are not. To be consistent in considering three neighbouring cells in the above equation, a further modification is needed:

$$\begin{aligned} P_{1x} = \mathcal{N}\Bigg\{ & pP_{1x}\left[\frac{P_{xx} + P_{x1} + P_{x2}(1-\mathcal{W})}{P_{xx} + P_{x1} + P_{x2}}\right] + P_{2x}\left[\frac{pP_{x1} + pP_{x2}(1-\mathcal{W})}{P_{xx} + P_{x1} + P_{x2}}\right] \\ & + P_{11}\left[\frac{(1-p)P_{1x}}{P_{1x} + P_{11}}\right] + P_{21}\left[\frac{(1-p)P_{1x}}{P_{1x} + P_{11}}\right]\Bigg\} \end{aligned} \tag{8}$$

where the GoE states implicitly included in the first term of the right-hand side are eliminated. Similarly, for variables P_{xx}, P_{x1}, P_{x2}, P_{11}, P_{2x} and P_{21}, the corresponding equations should

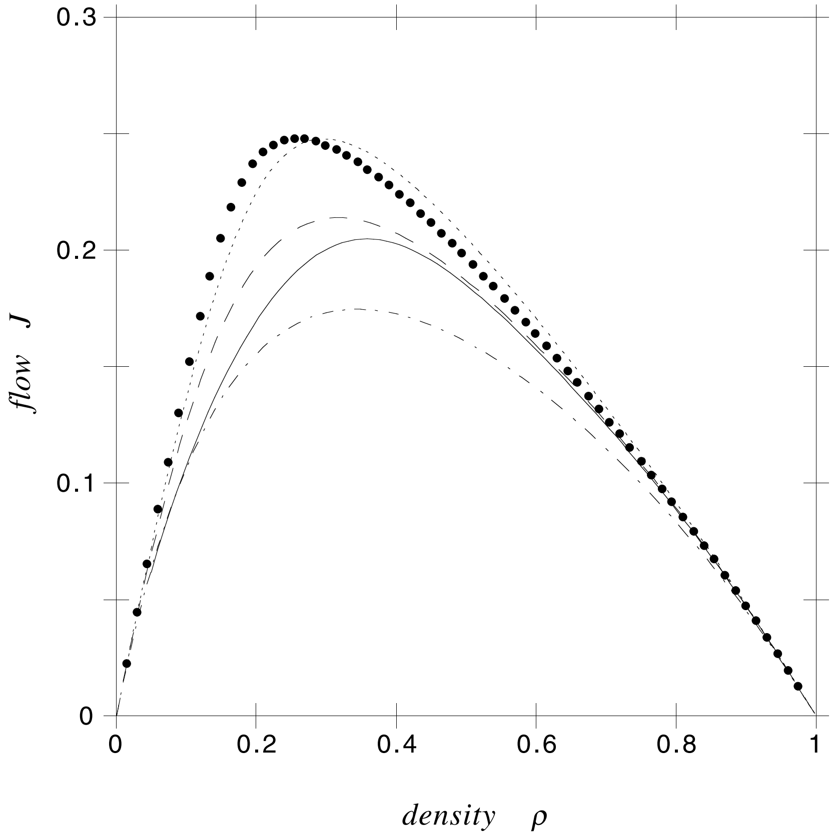

Figure 1. Fundamental diagram, flow J versus density ρ, for $v_{\max} = 2$ and $p = 0.5$. The solid line is the result of combing the cluster approximation with the paradisical mean-field theory. The dash–dotted, dashed and dotted curves are the results for the simple mean-field theory, cluster approximation and paradisical mean-field theory, respectively.

also be replaced (see the appendix). It is interesting to note that without the normalization, i.e. $\mathcal{N} = 1$, no solution of P_{ij} can be found. With the normalization $\mathcal{N}$ as a new variable, a unique solution can be determined for P_{ij} and $\mathcal{N}$. Numerically, an iteration scheme is employed with the initial values taken from P_{ij} of the two-cluster approximation and $\mathcal{N} = 1$. Then the flow becomes

$$J = (1-p)P_{1x}\left[\frac{P_{xx} + P_{x1} + P_{x2}(1-\mathcal{W})}{P_{xx} + P_{x1} + P_{x2}}\right] + P_{2x}\left[\frac{(2-p)P_{xx} + (1-p)P_{x1} + (1-p)P_{x2}(1-\mathcal{W})}{P_{xx} + P_{x1} + P_{x2}}\right]. \qquad (9)$$

The results are shown in figure 1. At first sight, this result is surprising. The flow is underestimated compared with the result of the two-cluster approximation. In this calculation, we eliminate all the GoE states: $\langle\langle 1x2\rangle\rangle$, $\langle\langle 2x2\rangle\rangle$ and $\langle\langle 1xx2\rangle\rangle$. There are no further elementary GoE states for clusters up to ten cells [8]. We have studied the effect of eliminating these GoE states separately. The results are similar. The flow is in between the results for the two-cluster approximation and simple mean-field theory (one-cluster approximation).

4. Discussion

In this Letter, a modified mean-field theory is framed to adjoin two successful mean-field theories: the cluster approximation and paradisical mean-field theory. However, the results do not make an improvement in describing the fundamental diagram. Various types of mean-field theory are all based on microscopic considerations and probability interpretation. In the cluster approximation, conditional probabilities are applied systematically. An improved result is obtained with a larger size of cluster considered, though the calculations are quite involved and the nonlinear system of equations can only be solved numerically even for relatively small cluster size.

In contrast, the probability interpretation has not been treated rigorously in the paradisical mean-field theory. The terms corresponding to the GoE states are deleted quite arbitrarily and then a new factor is introduced to renormalize the probability. Starting with the simple mean-field theory, i.e. in the one-cluster approximation, such a scheme leads to a considerable increase of the flow. Basically, the success of the paradisical mean-field theory lies in the elimination of nearest-neighbouring GoE states. These states have more weighting on the lower speed. Thus a renormalization will shift weighting toward a higher speed and the flow will increase considerably. Further eliminating the GoE states involving three or four neighbouring cells has only negligible effects. In contrast, when we start with the two-cluster approximation, the effects of nearest-neighbouring GoE states have already been included. We only have to consider the GoE states beyond the nearest-neighbouring cells. The GoE states of nearest-neighbouring cells involve a moving car, while those beyond the nearest-neighbouring cells involve a car moving at its highest speed. As these states have more weighting on the higher speed, the elimination and renormalization will shift weighting back to a lower speed and the flow will decrease, as shown in figure 1. It is interesting to note that eliminating the nearest-neighbouring GoE states will increase the flow more effectively in the paradisical mean-field theory than in the two-cluster approximation, while the probability interpretation is taken more rigorously in the two-cluster approximation. Thus we simply conclude that the success of paradisical mean-field theory is accidental. Such an approach cannot be improved systematically as the cluster approximation can.

Appendix

To be complete, we list the six equations for variables P_{xx}, P_{x1}, P_{x2}, P_{11}, P_{2x} and P_{21}, respectively:

$$
\begin{aligned}
P_{xx} = \mathcal{N}\Bigg\{ & P_{xx}\left[\frac{pP_{1x} + P_{xx}\left(\frac{P_{xx}+P_{1x}+pP_{2x}}{P_{xx}+P_{1x}+P_{2x}}\right)}{P_{xx}+P_{1x}+P_{2x}}\right] + P_{2x}\left[\frac{qP_{xx}}{P_{xx}+P_{x1}+P_{x2}}\right] \\
& + P_{x2}\left[\frac{P_{2x}\left(\frac{P_{xx}+qP_{x1}+qP_{x2}\mathcal{V}}{P_{xx}+P_{x1}+P_{x2}}\right)}{P_{2x}+P_{21}}\right]\left[\frac{pP_{1x}\mathcal{V} + pP_{2x}\mathcal{V} + P_{xx}\left(\frac{P_{xx}+P_{1x}\mathcal{V}+pP_{2x}}{P_{xx}+P_{1x}+P_{2x}}\right)}{P_{xx}+P_{1x}+P_{2x}}\right] \\
& + P_{x1}\left[\frac{qP_{1x}}{P_{1x}+P_{11}}\right]\left[\frac{pP_{1x} + pP_{2x} + P_{xx}\left(\frac{P_{xx}+P_{1x}+pP_{2x}}{P_{xx}+P_{1x}+P_{2x}}\right)}{P_{xx}+P_{1x}+P_{2x}}\right]\Bigg\}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
P_{x1} = \mathcal{N}\Bigg\{ & P_{x2}\left[\frac{P_{21} + P_{2x}\left(\frac{pP_{x1}+pP_{x2}\mathcal{V}}{P_{xx}+P_{x1}+P_{x2}}\right)}{P_{2x}+P_{21}}\right]\left[\frac{pP_{1x}\mathcal{V} + pP_{2x}\mathcal{V} + P_{xx}\left(\frac{P_{xx}+P_{1x}\mathcal{V}+pP_{2x}}{P_{xx}+P_{1x}+P_{2x}}\right)}{P_{xx}+P_{1x}+P_{2x}}\right] \\
& + P_{x1}\left[\frac{pP_{1x}+P_{11}}{P_{1x}+P_{11}}\right]\left[\frac{pP_{1x} + pP_{2x} + P_{xx}\left(\frac{P_{xx}+P_{1x}+pP_{2x}}{P_{xx}+P_{1x}+P_{2x}}\right)}{P_{xx}+P_{1x}+P_{2x}}\right]\Bigg\}
\end{aligned}
\tag{11}
$$

$$P_{x2} = \mathcal{N}\left\{ P_{xx}\left[\frac{qP_{2x}}{P_{xx}+P_{1x}+P_{2x}}\right] + qP_{1x}\left[\frac{P_{xx}+P_{x1}+P_{x2}\mathcal{V}}{P_{xx}+P_{x1}+P_{x2}}\right] + P_{2x}\left[\frac{pP_{xx}+qP_{x1}+qP_{x2}\mathcal{V}}{P_{xx}+P_{x1}+P_{x2}}\right]\right\} \tag{12}$$

$$P_{11} = \mathcal{N}\left\{ P_{11}\left[\frac{pP_{1x}+P_{11}}{P_{1x}+P_{11}}\right] + P_{21}\left[\frac{pP_{1x}+P_{11}}{P_{1x}+P_{11}}\right]\right\} \tag{13}$$

$$P_{2x} = \mathcal{N}\left\{ P_{xx}\left[\frac{qP_{1x}+pP_{2x}+P_{xx}(\frac{qP_{2x}}{P_{xx}+P_{1x}+P_{2x}})}{P_{xx}+P_{1x}+P_{2x}}\right] + P_{x2}\left[\frac{P_{2x}(\frac{P_{xx}+qP_{x1}+qP_{x2}\mathcal{V}}{P_{xx}+P_{x1}+P_{x2}})}{P_{2x}+P_{21}}\right]\left[\frac{qP_{1x}\mathcal{V}+qP_{2x}\mathcal{V}+P_{xx}(\frac{qP_{2x}}{P_{xx}+P_{1x}+P_{2x}})}{P_{xx}+P_{1x}+P_{2x}}\right] + P_{x1}\left[\frac{qP_{1x}}{P_{1x}+P_{11}}\right]\left[\frac{qP_{1x}+qP_{2x}+P_{xx}(\frac{qP_{2x}}{P_{xx}+P_{1x}+P_{2x}})}{P_{xx}+P_{1x}+P_{2x}}\right]\right\} \tag{14}$$

$$P_{21} = \mathcal{N}\left\{ P_{x2}\left[\frac{P_{21}+P_{2x}(\frac{pP_{x1}+pP_{x2}\mathcal{V}}{P_{xx}+P_{x1}+P_{x2}})}{P_{2x}+P_{21}}\right]\left[\frac{qP_{1x}\mathcal{V}+qP_{2x}\mathcal{V}+P_{xx}(\frac{qP_{2x}}{P_{xx}+P_{1x}+P_{2x}})}{P_{xx}+P_{1x}+P_{2x}}\right] + P_{x1}\left[\frac{pP_{1x}+P_{11}}{P_{1x}+P_{11}}\right]\left[\frac{qP_{1x}+qP_{2x}+P_{xx}(\frac{qP_{2x}}{P_{xx}+P_{1x}+P_{2x}})}{P_{xx}+P_{1x}+P_{2x}}\right]\right\} \tag{15}$$

(16)

where $q = 1 - p$ and $\mathcal{V} = 1 - \mathcal{W}$.

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